Generalized Inference: Applications to Mixed Linear Models

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TERMINOLOGY

- Tsui & Weerahandi (1989) introduced the terms
  - Generalized P-value (GPV)
  - Generalized Test Variable (GTV).
- Weerahandi (1993) introduced the terms
  - Generalized Pivotal Quantities (GPQ)
  - Generalized Confidence Intervals (GCI).
- The term Generalized Inference (GI) refers to inference procedures that are based on the above concepts.

TERMINOLOGY

- Using this approach one can develop hypothesis tests and confidence interval procedures for certain classes of parametric models when exact pivotal quantities are not available.
- In particular, the approach leads to "good" inference procedures in (balanced) normal mixed linear models. But the method is more generally applicable.

OUTLINE

1. What is a GPQ? How it leads to a GCI?
2. What is a GTV? How it leads to a GPV?
3. Simple Examples of GPQs, GCIs, GTVs, and GPVs.
4. Recipe for Constructing GPQs
5. Examples (a) Exact Methods (b) Approximate Methods
6. Generalized Inference in Balanced Mixed Linear Models
7. Some nonstandard applications
8. References
9. Remarks: Historical connections
   Issues in unbalanced situations
   Extensions

Notation

$\boldsymbol{D}$ = observable data vector

$\boldsymbol{d}$ = observed value of $\boldsymbol{D}$

$\boldsymbol{\xi}$ = vector of parameters

τ = $h(\boldsymbol{\xi})$, a scalar function of $\boldsymbol{\xi}$ about which inference is to be made (test or confidence interval)

WLOG we can assume that $\boldsymbol{\xi} = (\tau, \boldsymbol{\zeta})$ where τ is the scalar parameter of interest and ζ is a vector of nuisance parameters

Generalized Pivotal Quantity

$R = R(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi})$, a function of $\boldsymbol{D}$, $\boldsymbol{d}$, and $\boldsymbol{\xi}$, is called a **Generalized Pivotal Quantity** if it satisfies the following two properties (Weerahandi, 93):

1. Distribution of R is free of unknown parameters.
2. The **observed pivotal** quantity $r_{obs} = r = R(\boldsymbol{d}; \boldsymbol{d}, \boldsymbol{\xi})$ depends on $\boldsymbol{\xi}$ only through τ.

Generalized Confidence Interval

An equal tailed $1 - \alpha$ GCI (confidence set) for τ is obtained as the set

$$\{\tau \mid R_{\alpha/2} \leq r \leq R_{1-\alpha/2}\}$$

Often the confidence set reduces to an interval $[L, U]$.

Example: One-Sample Problem

$Y_1, \ldots, Y_n \sim$ iid $N(\mu, \sigma^2)$

$y_1, \ldots, y_n$ are observed values

$\bar{Y}$ = sample mean, S = sample standard deviation.

$\bar{y}, s$ the corresponding realized values (known constants)

Usual pivotal quantity: $T = \dfrac{\bar{Y} - \mu}{S/\sqrt{n}}$

Realized value: $t = \dfrac{\bar{y} - \mu}{s/\sqrt{n}}$.

Classical t-interval is $\{\mu | t_{\alpha/2} \leq t \leq t_{1-\alpha/2}\}$

i.e., $\left\{ \mu \middle| \; \bar{y} - t_{1-\alpha/2:n-1} \left(\dfrac{s}{\sqrt{n}} \right) \leq \mu \leq \bar{y} - t_{\alpha/2:n-1} \left(\dfrac{s}{\sqrt{n}} \right) \right\}$

Example: One-Sample Problem

Define $R = \bar{y} - \left(\frac{s}{S}\right)(\bar{Y} - \mu) = \bar{y} - \left(\frac{s}{\sqrt{n}}\right)\frac{\bar{Y} - \mu}{S/\sqrt{n}}$.

R is said to be a **Generalized Pivotal Quantity** for μ.

The **observed pivotal** is r = μ.

A GCI for μ is $\{\mu | R_{\alpha/2} \leq \mu \leq R_{1-\alpha/2}\}$

Note $\quad R = \bar{y} - \left(\frac{s}{\sqrt{n}}\right) T \quad$ so, $R_\gamma = \bar{y} - \left(\frac{s}{\sqrt{n}}\right) T_{1-\gamma}$.

$\{\mu | R_{\alpha/2} \leq r \leq R_{1-\alpha/2}\} = \{\mu | t_{\alpha/2} \leq \frac{\bar{y} - \mu}{s/\sqrt{n}} \leq t_{1-\alpha/2}\}$.

Thus, the GCI is the same as the classical t-interval.

Generalized Test Variable and GPV

We wish to test
$H_0 : \theta \leq \theta_0$ versus $H_a : \theta > \theta_0$

$T = T(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi})$ is called a **GTV** if it satisfies:

1. The distribution of T depends on $\boldsymbol{\xi}$ only through θ. In particular, it is completely determined when θ is specified.
2. The **observed value** of the test variable $t = t_{obs} = T(\boldsymbol{d}; \boldsymbol{d}, \boldsymbol{\xi})$ is free of unknown parameters.
3. For fixed $\boldsymbol{d}$, ζ, and t^*, $Pr[T(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi}) > t^*]$ is a **nondecreasing** function of θ.
4. $GPV = Pr[T(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi}) > t_{obs} \,|\, \theta = \theta_0]$

Example: One-Sample Problem - GTV

Define $V = \mu - \bar{y} + \left(\frac{s}{S}\right)(\bar{Y} - \mu)$

V is said to be a **Generalized Test Variable** (GTV) for testing
$H_0 : \mu \leq \mu_0$ *versus* $H_0 : \mu > \mu_0$.

The **observed test variable** is v = 0.

GPV $\stackrel{def}{=} P\left[V \geq v | \mu = \mu_0\right]$

Here $GPV = P\left[\frac{\bar{Y} - \mu}{S/\sqrt{n}} \geq \frac{\bar{y} - \mu_0}{s/\sqrt{n}}\right] = P\left[T \geq t_0\right]$
where t_0 = the usual computed t-statistic.

So, in this example, GPV = Ordinary P-value

A Recipe for Construcing GPQs

Joint distribution of the data vector $\boldsymbol{D}$ is indexed by a k-dimensional parameter $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_k) \in \boldsymbol{\Omega} \subseteq R^k$.

$\tau = h(\boldsymbol{\xi})$, a scalar function for which inference is required.

Assume

(a) There exist a mapping $\boldsymbol{f} : R^k \times R^k \rightarrow R^k$, such that, $(U_1, \ldots, U_k) = \boldsymbol{U} = \boldsymbol{f}(\boldsymbol{D}; \boldsymbol{\xi})$ has a joint distribution free of $\boldsymbol{\xi}$.

(b) For each $\boldsymbol{D}$, there exists a mapping $\boldsymbol{g}(\boldsymbol{D}; \cdot) : R^k \rightarrow R^k$ such that

$$\boldsymbol{g}(\boldsymbol{D}; \boldsymbol{U}) = \boldsymbol{g}(\boldsymbol{D}; \boldsymbol{f}(\boldsymbol{D}; \boldsymbol{\xi})) = (g_1(\boldsymbol{D}; \boldsymbol{U}), \cdots, g_k(\boldsymbol{D}; \boldsymbol{U})) = \boldsymbol{\xi}.$$

Recipe-continued

Define

$$R = R(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi}) = h(\boldsymbol{g}(\boldsymbol{d}; \boldsymbol{f}(\boldsymbol{D}; \boldsymbol{\xi}))) = h(\boldsymbol{g}(\boldsymbol{d}; \boldsymbol{U}))$$

1. R is a GPQ for $\tau = h(\boldsymbol{\xi})$.
2. $R_{\alpha/2} \leq \tau \leq R_{1-\alpha/2}$ is an equal-tailed 2-sided GCI for τ. (One-sided Generalized Bounds obtained in an obvious manner).
3. $T = T(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi}) = h(\boldsymbol{\xi}) - R = \tau - R$, is a Generalized Test Variable for testing $H_0 : \tau \leq \tau_0$ versus $H_a : \tau > \tau_0$.
4. $GPV = Pr\left[T(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi}) \geq 0 \mid \tau = \tau_0\right]$

Balanced Mixed Linear Models

In many Balanced Mixed Linear Models the ANOVA sums of squares and sample cell means form a set of complete sufficient statistics. For instance, this is the case when the model is saturated.

The joint distribution of this set has a simple structural representation.

The recipe may be applied to produce tests and confidence intervals for functions of the model parameters.

Simulation studies show that these work well.

Some general theoretical results exist that provide insight into why these methods perform well.

General Setting

- Suppose $SS_1, \ldots, SS_q$ are the sums of squares in the ANOVA table corresponding to the random effects. Also suppose $\hat{\boldsymbol{\beta}}$ is the vector of estimates of the cell means generated by the fixed factors.
- Denote the cell means by $\beta_1, \ldots, \beta_p$.
- Denote the expected Mean Squares (EMS) by $\theta_1, \ldots, \theta_q$
- $\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\Sigma} = \theta_1 \boldsymbol{V}_1 + \cdots + \theta_q \boldsymbol{V}_q$$

and $\boldsymbol{V}_i$ are matrices of known constants.

GPQs

- The θ_i admit GPQs of the following form:

$$R_{\theta_i} = \frac{ss_i}{U_i} = \frac{(ss_i)(\theta_i)}{SS_i}, \qquad i = 1, \ldots, q$$

  where $U_i \sim \chi^2_{\nu_i}$ (jointly independent).

- $\boldsymbol{\beta}$ admits a GPQ given by $\quad \boldsymbol{R}_{\boldsymbol{\beta}} = \boldsymbol{b} - \boldsymbol{R}_{\boldsymbol{C}}\boldsymbol{Z} \quad$ where $\boldsymbol{b}$ is the observed value of $\hat{\boldsymbol{\beta}}$ and $\boldsymbol{R}_{\boldsymbol{C}}$ is the Cholesky factor (lower triangular) of the matrix $\boldsymbol{R}_{\boldsymbol{\Sigma}} = R_{\theta_1}\boldsymbol{V}_1 + \cdots + R_{\theta_q}\boldsymbol{V}_q$.

- Let τ be any function, say $f(\boldsymbol{\theta}, \boldsymbol{\beta})$ of the model parameters for which a confidence interval is sought. Then $R_\tau = f(\boldsymbol{R}_{\boldsymbol{\theta}}, \boldsymbol{R}_{\boldsymbol{\beta}})$ is a GPQ for τ.

One-way Nested Random Model

$X_{ij} = \mu + A_i + e_{ij}$. $\quad i = 1, \ldots, a;\ j = 1, \ldots, n$.

$A_i \sim N(0, \sigma_A^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$.

All random variables jointly independent.

We want a confidence interval for σ_A^2.
Methods that have appeared in the literature:

- Tukey-Williams
- Moriguti-Bulmer
- Howe
- Graybill-Wang

Example

$$Z = \frac{\bar{X} - \mu}{\sqrt{\frac{n\sigma_A^2 + \sigma_e^2}{na}}}, \qquad U_1 = \frac{SSw}{\sigma_e^2}, \qquad U_2 = \frac{SSb}{\sigma_e^2 + n\sigma_A^2}$$

$$Z \sim N(0,1), \qquad U_1 \sim \chi^2_{a(n-1)} \qquad U_2 \sim \chi^2_{a-1}$$

$$\mu = \bar{X} - Z\sqrt{\frac{SSb}{naU_2}},$$

$$\sigma_e = \sqrt{\frac{SSw}{U_1}} \qquad \sigma_A = \sqrt{max\left\{0, \frac{SSb}{nU_2} - \frac{SSw}{nU_1}\right\}}$$

GPQ

A GPQ for σ_A^2 is given by

$$\begin{aligned} R &= max\left\{0, \frac{ssb}{nU_2} - \frac{ssw}{nU_1}\right\} \\ &= \max\left\{0, (n\sigma_A^2 + \sigma_e^2)\frac{ssb}{nSSb} - \sigma_e^2\frac{ssw}{nSSw}\right\} \end{aligned}$$

EXAMPLE

Weights of bottles selected from filling machines

Machines			
1	2	3	4
14.23	16.46	14.98	15.94
14.96	16.74	14.88	16.07
14.85	15.94	14.87	14.91

Type 3 Analysis of Variance

Source	**DF**	**Sum of Squares**	**Mean Square**	**EMS**
machine	3	5.329425	1.776475	$\sigma_E^2 + 3\sigma_A^2$
Residual	8	1.454600	0.181825	σ_E^2

Confidence Interval for σ_A^2

The GPQ for σ_A^2 is

$$
\begin{aligned}
R &= max\left\{0, \frac{ssb}{nU_2} - \frac{ssw}{nU_1}\right\} \\
&= max\left\{0, \frac{5.329425}{3U_2} - \frac{1.4546}{3U_1}\right\}
\end{aligned}
$$

Graybill-Wang interval for σ_A^2 is [0.107, 8.16]

GCI for σ_A^2 is [0.09624, 8.19219] by simulation

GCI for σ_A^2 is [0.09605, 8.152] by numerical evaluation

Other methods work well also

$\sigma_A^2 + \sigma_E^2$

$$\sigma_A^2 + \sigma_E^2 = \frac{1}{n}(n\sigma_A^2 + \sigma_E^2) + \frac{n-1}{n}\sigma_E^2.$$

$$R = max\left\{0, \frac{ssb}{nU_2} + \frac{(n-1)ssw}{nU_1}\right\} =$$
$$max\left\{0, \frac{5.329425}{3U_2} + \frac{2(1.4546)}{3U_1}\right\}.$$

Welch-Satterthwaite, Graybill-Wang work well. GCI is competitive.

	GCI	Graybill-Wang
Lower bound	0.313	0.306
Upper bound	8.403	8.360

Exact calculation for GCI gives [0.31241,8.398] (maple)
GCI for σ_A^2/σ_E^2 coincides with the usual exact interval based on the ratio MS_B/MS_E.

Brand 1	Machine 1	15.66	15.66	15.70	15.70	15.68	15.70
	2	15.69	15.71	15.68	15.72	15.71	15.72
	3	15.73	15.68	15.73	15.71	15.67	15.72
	4	15.72	15.73	15.74	15.74	15.73	15.75
Brand 2	Machine 1	15.78	15.80	15.78	15.79	15.78	15.79
	2	15.78	15.76	15.76	15.77	15.76	15.77
	3	15.76	15.80	15.78	15.78	15.79	15.78
	4	15.77	15.80	15.78	15.78	15.77	15.78

ANOVA for EXAMPLE

BRAND-1

Source	DF	Sum of Squares	Mean Square
machine	3	0.008083	0.002694
Residual	20	0.007167	0.000358

BRAND-2

Source	DF	Sum of Squares	Mean Square
machine	3	0.001312	0.000437
Residual	20	0.002150	0.000108

$(\sigma^2_{A1} + \sigma^2_{E1})/(\sigma^2_{A2} + \sigma^2_{E2})$

$$R = \frac{\frac{ssb1}{n_1 U_{21}} + \frac{(n_1 - 1)ssw1}{n_1 U_{11}}}{\frac{ssb2}{n_2 U_{22}} + \frac{(n_2 - 1)ssw2}{n_2 U_{12}}} = \frac{\frac{0.008083}{6U_{21}} + \frac{5(0.007167)}{6U_{11}}}{\frac{0.001312}{6U_{22}} + \frac{5(0.002150)}{6U_{12}}}$$

	GCI	Burdick-Graybill
90% Lower bound	1.068	1.03
90% Upper bound	24.412	***

$$\sigma_A^2 + \sigma_B^2 + \sigma_E^2$$

Cage	1			2			3			4		
Mosquito	1	2	3	1	2	3	1	2	3	1	2	3
	58.5	59.5	77.8	80.9	84.0	83.6	70.1	68.3	69.8	69.8	56.0	54.5
	50.7	49.3	63.8	65.8	56.6	57.5	77.8	79.2	69.9	69.2	62.1	64.5

ANOVA

Source	DF	Sum of Squares	Mean Square
cage	2	665.675833	332.837917
mosquito(cage)	9	1720.677500	191.186389
Residual	12	15.620000	1.301667

	GCI	Burdick-Graybill
95% One sided Lower bound	63.3212	65.5
95% One sided Upper bound	1763.68	1724

Two way Crossed Random Model

$$Y_{ijk} = \mu + A_i + B_j + (AB)_{ij} + E_{ijk}$$

i = Mice Strain (5); j = Day (6); k = Mice (5)

Source	Df	SS	MS
Strain	4	0.3680	0.0920
Days	5	0.0505	0.0101
Interaction	20	0.1040	0.0052
Error	120	0.4080	0.0034

(Weir, 1949)

Need CI for $\sigma_A^2/(\sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2 + \sigma_E^2)$.

	GCI	Leiva-Graybill
90% Lower bound	0.157	0.196
90% Upper bound	0.770	0.814

A Mixed Model Example

Source	Df	SS	MS
Temperature (A)	2	616.78	308.39
Speed (B)	3	175.56	58.52
Pressure (C)	2	5.04	2.52
AB	6	809.46	134.91
AC	4	179.08	44.77
BC	6	115.56	19.26
ABC	12	231.12	19.26
Error	36	1248.12	34.67

(Montgomery, 1984)

Temp FIXED, Speed RANDOM

Pressure RANDOM

Need CI for $\mu_1 - \mu_2$

$V(\widehat{\mu_1 - \mu_2}) = (\theta_{AB} + \theta_{AC} - \theta_{ABC})/24$

$= (6\sigma^2_{AB} + 8\sigma^2_{AC} + 2\sigma^2_{ABC} + \sigma^2_e)/24$

No exact interval available

	GCI	Banerjee (1960)
95% Lower bound	-1.146	-3.28
95% Upper bound	17.742	19.9

An Unbalanced Example

- An artifact measured by each of k labs (or, k methods)
- Lab i makes n_i measurements
- Data are $Y_{ij}, j = 1, \cdots, n_i, i = 1, \cdots, k$.
- Model is: $\quad Y_{ij} = \mu_i + e_{ij}$
- μ = the true value.
- $\mu_i - \mu = b_i$ is the “bias" of lab i.
- Parameter of interest is μ.
- Estimate μ using combined information from all labs.
- $e_{ij} \sim N(0, \sigma_i^2)$.

Three Models

- **Model-1:** One-way random effects model with unequal sample sizes and heterogeneous variances. (See Rukhin and Vangel (1998), Vangel and Rukhin (1999), Rukhin, Biggerstaff, and Vangel (2000), Paule and Mandel (1971, 1982) – Large sample methods).
- **Model-2: (Bounded Bias Model)** b_i are in the (known) interval $[m_i, M_i]$. (Eberhardt, Reeve, and Spiegelman (1989).)
  Eberhardt et al. derived a minimax MSE linear estimator for μ. Proposed approximate CIs.
- **Model-3: (GUM model)** b_i have *known* distribution, say F_i. (see, Expression of Uncertainty in Measurement (ISO GUM) (1995); the distributions are referred to as *type-B* distributions.)

GPQ in Model 3

$$R^*(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\theta}) = \bar{y}_{\boldsymbol{W}} - \bar{b}_{\boldsymbol{W}} - Z^* \left(\sum_{i=1}^{k} n_i Q_i / ss_i \right)^{-1/2}$$

where,

$$\bar{y}_{\boldsymbol{W}} = \sum_{i=1}^{k} W_i \bar{y}_i / \sum_{i=1}^{k} W_i, \quad \bar{b}_{\boldsymbol{W}} = \sum_{i=1}^{k} W_i b_i / \sum_{i=1}^{k} W_i,$$

$U_i = \bar{Y}_i - b_i$, $W_i = \dfrac{n_i SS_i}{\sigma_i^2 \; ss_i}$

$\tau_0^2 = \dfrac{1}{w_1 + \cdots + w_k}$

$Z^* = (\bar{U}_{\boldsymbol{w}} - \mu)/\tau_0 \sim N(0,1)$, and

$Q_i = SS_i/\sigma_i^2 \sim \chi^2_{n_i-1}$, $i = 1, \ldots, k$.

Example

Zinc (μg/g) in non-fat milk powder

Method	n_i	$\bar{y}_i$	s_i	M_i
1	8	45.21	1.68	5.880
2	12	46.63	0.47	0.466
3	22	46.26	0.82	0.927
4	8	47.05	1.44	0.230

Distribution	Lower Bound	Upper bound
Uniform $[-M_i, M_i]$	45.85	47.05
$N(0, M_i/3)$	46.03	46.86

Example

Tolerance bounds for the distribution of true values when there are measurement errors

$X_{ij} = \mu + A_i + e_{ij}$. $\quad i = 1, \ldots, a;\ j = 1, \ldots, n$.

$A_i \sim N(0, \sigma_A^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$.

All random variables jointly independent.

Need a γ-content, $1 - \alpha$ confidence, upper tolerance-bound for the distribution of $\mu + A_i$, i.e., for $N(\mu, \sigma_A^2)$.

This is equivalent to an upper confidence bound for $\mu + z_\gamma \sigma_A$.

Wang and Iyer (1994, Technometrics) have discussed this problem.

Example-continued

$$Z = \frac{\bar{X} - \mu}{\sqrt{\frac{n\sigma_A^2 + \sigma_e^2}{na}}}, \qquad U_1 = \frac{SSw}{\sigma_e^2}, \qquad U_2 = \frac{SSb}{\sigma_e^2 + n\sigma_A^2}$$

$$Z \sim N(0,1), \qquad U_1 \sim \chi^2_{a(n-1)} \qquad U_2 \sim \chi^2_{a-1}$$

$$\mu = \bar{X} - Z\sqrt{\frac{SSb}{naU_2}},$$

$$\sigma_e = \sqrt{\frac{SSw}{U_1}} \qquad \sigma_A = \sqrt{max\left\{0, \frac{SSb}{nU_2} - \frac{SSw}{nU_1}\right\}}$$

GPQ

A GPQ for $\theta = \mu + z_\gamma \sigma_A$ is given by

$$
\begin{aligned}
R &= \bar{x} - Z\sqrt{\frac{ssb}{naU_2}} + z_\gamma \sqrt{max\left\{0, \frac{ssb}{nU_2} - \frac{ssw}{nU_1}\right\}} \\
&= \bar{x} - (\bar{X} - \mu)\sqrt{\frac{ssb}{SSb}} \\
&\quad + z_\gamma \sqrt{\max\left\{0, (n\sigma_A^2 + \sigma_e^2)\frac{ssb}{nSSb} - \sigma_e^2 \frac{ssw}{nSSw}\right\}}
\end{aligned}
$$

Historical Connections

- GPV and GCI are intimately related to Fisher's Fiducial Inference and Fraser's Structural Inference.
- Fiducial Inference and Structural Inference allow one to make probability statements about model parameters somewhat akin to Bayesian Posterior Distributions for parameters, but do not rely on any prior distributions for the parameters – (Savage: "..eat the Bayesian omelette without breaking the Bayesian Egg").
- The probability statements are EXACT in their own setting but do not seem to have satisfactory frequentist interpretations. This led to quite a controversy over the use of Fiducial/Structural methods during the mid and latter part of the 20th century.

Historical Connections

- GPVs and GCIs have a fiducial/structural flavor to them but Tsui and Weerahandi have put forward these ideas in a frequentist context. Although the inference is only APPROXIMATE in all but the simplest problems, Weerahandi refers to them as EXACT methods. The exactness properties of the procedures refers to their own setting and not to the usual frequentist setting.
- Methods for developing GPQs and GTVs were discussed by Iyer and Patterson (2002) where they used Fraser's structural distributions for parameters to construct GPQs and GTVs. No other general methods appear to be available.

Historical Connections

- Andy Chang (2001) proposed the method of SURROGATE VARIABLES and derived some confidence interval procedures for a class of mixed models. His approach is essentially an application of Fraser's structural inference to construct GPQs for this class of problems.
- It is not clear whether generalized inference is EQUIVALENT to structural inference.
- IGNORING philosophical issues related to the meanings of fiducial or structural probability statements, if one examines frequentist properties of these methods, more often than not, they lead to competing procedures and often methods better than what is currently available. In many cases, the methods can be shown to also be obtainable using Bayesian arguments.

Structural Distributions – Basic Idea

Let $Y \sim N(\mu, 1)$. Then Y has the structural representation

$$Y = \mu + Z$$

where $Z \sim N(0, 1)$.
Suppose an observed value of Y is $y = 2$. We infer that a value z for Z has been realized such that

$$2 = \mu + z$$

If we want to know how plausible it is that $\mu = 10$, this is equivalent to asking how plausible it is that $z = -8$. The known distribution of Z helps us assess this. Thus, the distribution of Z **induces** a distribution on μ (called 'Structural Distribuion' by Fraser). In this example the induced distribution on μ is $N(2, 1)$. We may write

Structural Distributions - 2

$Y_1, \cdots, Y_n$ iid sample from $N(\mu, \sigma^2)$.

Sufficient statistics: $\overline{Y}$, S^2.

Sructural representation:

$$\overline{Y} = \mu + \frac{\sigma}{\sqrt{n}} Z \qquad\qquad \frac{(n-1)S^2}{\sigma^2} = U$$

Substitute observed values ($\overline{y}$, s^2) for $\overline{Y}$, S^2 and get:

$$\mu = \overline{y} - \left(\frac{s}{\sqrt{n}}\right)\left(\frac{Z}{\sqrt{U/(n-1)}}\right) = \overline{y} - \left(\frac{s}{\sqrt{n}}\right) T_{n-1}$$

Thus, μ and σ may be thought to have a joint structural distribution induced by the joint distribution of Z and U.

Structural Distributions - continued

Let $\tau_\gamma = \mu + Z_\gamma \sigma$, the γ^{th} percentile of the $N(\mu, \sigma^2)$ distribution. Suppose we want an upper confidence bound for τ_γ with confidence coefficient $1 - \alpha$.

Derive the structural distribution of τ_γ and use the γ^{th} percentile of this distribution as an upper confidence bound for τ_γ. Such a bound is often referred to as a γ-content, $1 - \alpha$ coverage, one-sided tolerance bound for the distribution $N(\mu, \sigma^2)$.

FACT: The structural approach results in exactly the same tolerance bound as does the classical method based on a noncentral t-distribution.